TEZPUR UNIVERSITY
Assignment Spring 2022
MMS 303 : Number Theory
Total Marks: 30

The figures in the right-hand margin indicate marks for the individual question.
All questions are compulsory.
Answers should be concise and entire answer to a question should be together. State assumptions wherever made.

1. Assume that $\gcd(a, b) = 1$. Prove the following: **2+2**
   (a) $\gcd(a + b, a^2 + b^2) = 1$ or 2.
   (b) $\gcd(a + b, a^2 - ab + b^2) = 1$ or 3.
2. The $R_1 = 1$, $R_2 = 11$, $R_3 = 111$, $R_4 = 1111$, and so on. Prove the following assertions. **2+2+2**
   (a) If $m|n$, then $R_n|R_m$.
   (b) If $d|R_n$ and $d|R_m$, then $d|R_{n+m}$.
   (c) If $\gcd(n, m) = 1$, then $\gcd(R_n, R_m) = 1$.
3. If p and $p^2 + 8$ are both prime numbers, prove that $p^3 + 4$ is also prime. **3**
4. Let $\omega(n)$ denote the number of distinct prime divisors of $n > 1$, with $\omega(1) = 0$. For instance, $\omega(360) = \omega(2^3 \cdot 3^2 \cdot 5) = 3$.
   (a) Show that $2^{\omega(n)}$ is a multiplicative function. **2**
   (b) For a positive integer n, establish the formula **3**

   $$\tau(n^2) = \sum_{d|n} 2^{\omega(d)},$$

   where $\tau(n)$ denotes the number of positive divisors of n.
5. For any positive integer n, prove that $\sum_{d|n}(\tau(d))^3 = \left(\sum_{d|n} \tau(d)\right)^2$, where $\tau(n)$ is as defined in the previous question. **2**
6. Obtain three consecutive integers, each having a square factor. **5**
7. Prove that 3 is a quadratic nonresidue of all primes of the form $2^{2n} + 1$, and all primes of the form $2^p - 1$, where p is an odd prime. **5**

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