ME 428: Finite Element Method
Lecture 4

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1. Introduction

In the last lecture, you have learnt how to obtain the variational form of a given boundary value problem. We also discussed the virtual work principle in reference to obtain the variational form particularly for structural problems. In the process of obtaining the variational form of the axial bar problem you have seen that at the end of the procedure we arrived at the condition of extremizing a functional along with the satisfaction of the boundary conditions. That functional is the variational form or the integral form of the boundary value problem. The integral form needs to be extremized to obtain the solution of the boundary value problem. This formulation is called **variational formulation**. In this lecture, we will discuss the physical interpretation of the variational form. We also discuss some problems.

2. Principle of minimum potential energy

We obtained the variational form of the axial bar problem as

$$I = \int_0^L \left\{ \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 - qu \right\} dx - Pu \big|_{x=L}. \tag{1}$$

Equation (1) has a physical interpretation. The quantity I in Eq. (1) represents the total potential energy of the bar.

The total potential energy of the bar consists of (a) the strain energy of the elastic deformation and (b) the potential possessed by the applied loads. The strain energy of a linear elastic body is given by

$$U = \int_V \frac{1}{2} \sigma \varepsilon \, dV, \tag{2}$$

where σ is the stress and ε is the strain. For the problem of axial bar

$$\varepsilon = \frac{du}{dx}, \tag{3}$$

$$\sigma = E \frac{du}{dx}. \tag{4}$$

Thus, in the present case,

$$U = \int_V \frac{1}{2} E \left(\frac{du}{dx} \right)^2 dV,$$

$$U = \int_0^L \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx, \tag{5}$$

where A is the cross sectional area and L is the length of the rod.

The work potential due to the axial distributed load $q(x)$ is given by

$$V_q = -\int_0^L qu\,dx. \tag{6}$$

The work potential due to the concentrated load at the extreme end of the bar is

$$V_P = -Pu\big|_{x=L}. \tag{7}$$

Thus, the total potential energy of the rod is given by

$$\Pi = U + V_q + V_P = \int_0^L \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx - \int_0^L qu\,dx - Pu\big|_{x=L}. \tag{8}$$

Equation (8) is the same expression that we obtained for the variational form I in Eq. (1). Therefore the variational formulation states the principle of minimum potential energy, which can be stated as

"*For conservative structural systems, of all the kinematically admissible deformations, those corresponding to the equilibrium state, extremize the total potential energy. If the extremum is a minimum, the equilibrium state is stable'.*

3. Strong and weak form of a problem representing a physical system

We can represent a physical system either as a boundary value problem consisting of a differential equation with boundary conditions or by an integral form known as the variational form if obtained by using the variational calculus. The boundary value problem, *i.e.*, the differential form of the physical system is called the **strong form**. It is called as the strong form, because it contains higher order derivatives. However, the integral form containing the lower order derivatives is called the **weak form**.
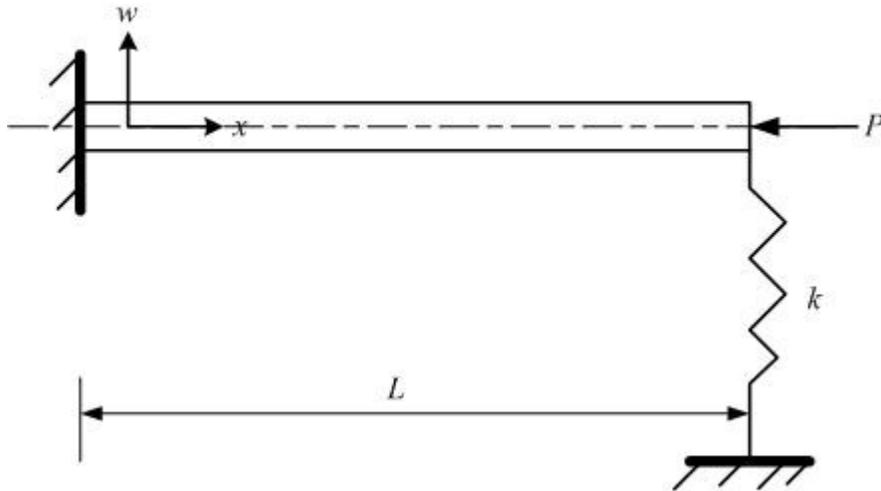
4. Exercise problems

(i) The functional governing static buckling of the column in Fig. 4.1 is given as

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx + \frac{1}{2}kw_L^2 \tag{9}$$

where
$$w\big|_{x=0} = 0 \quad \text{and} \quad \frac{dw}{dx}\bigg|_{x=0} = 0. \tag{10}$$

Derive the governing differential equation of the problem.

Fig. 4.1 Static buckling of a column

(ii) Consider the following boundary value problem

$$\frac{d^2u}{dx^2} + u = 1, \qquad 0 \le x \le 1 \tag{11}$$

with $u(0) = 0$ and $\frac{du}{dx} = 0$ at $x = 1$. Convert this problem into variational form.

(iii) The bending of a beam is governed by the following differential equation:

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) + \frac{dm_z}{dx} - q_y = 0 \tag{12}$$

with the following essential boundary conditions:

$$\text{at } x = 0, \ v = v^* \text{ and } \frac{dv}{dx} = \theta_z^* \tag{13}$$

and the following natural boundary conditions:

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$$\text{at } x = L, \ EI\frac{d^2v}{dx^2} = M_z^* \quad \text{and} \quad \frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) + m_z = -V_y^* \tag{14}$$

where m_z is the distributed moment per unit length about z-axis and q_y is the distributed force per unit length in y-direction. Obtain the variational form of the problem.