ME 428: Finite Element Method
Lecture 1

Dr. S. M. Kamal

1. Introduction

Finite element method is a numerical technique to solve boundary value problems. As an example, let us consider the following boundary value problem:

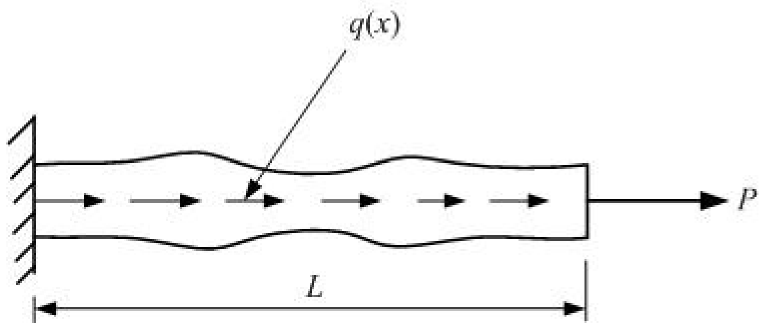

Fig. 1.1 Axial extension of a bar of variable cross-section

Consider the axial extension of a bar of variable cross-sectional area $A(x)$, fixed at one end as shown in Fig. 1. The length of the bar is L. The Young's modulus of the bar is E. The axial displacement of a cross-section at x, denoted by u(x) is governed by the following differential equation with two boundary conditions.

Differential Equation: $$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EA(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = q(x), \qquad 0 < x < L \tag{1}$$

Boundary conditions: $$\text{(i)}\ u = 0 \ \text{ at } \ x = 0, \tag{2}$$

$$\text{(ii)}\ EA(x)\frac{\mathrm{d}u}{\mathrm{d}x} = P \quad \text{at } x = L. \tag{3}$$

Now, the above boundary problem can be solved using finite element method (FEM). To solve this problem using FEM, you need follow a step by step procedure. The general steps involved in solving any boundary value problem are as follows:

Step 1: The governing differential equation of the problem is converted into an integral form. There are two techniques to convert differential equation into an integral form. They are— (i) **Variational Technique** and (ii) **Weighted Residual Technique**. In variational technique, the **calculus of variation** is used to obtain the integral form corresponding to the given differential equation. This integral needs to be minimized to obtain the solution of the problem. For structural mechanics problems, the integral form turns out to be the expression for the total potential energy of the structure. In weighted residual technique, the integral form is constructed as a weighted integral of the governing differential equation where the weight functions are known and arbitrary except that they satisfy certain boundary conditions. This integral form is set to zero to obtain the solution of the problem. For structural mechanics problems, if the weight function is considered as the virtual displacement, then the integral form becomes the expression of the virtual work of the structure.

Step 2: The domain of the problem is divided into a number of parts, called as **elements**. For one-dimensional (1-D) problems, the elements are the line segments having only length and no shape. For problems of higher dimensions, the elements have both the shape and size. For two-dimensional (2D) or axi-symmetric problems, the elements used are triangles, rectangles and quadrilateral having straight or curved boundaries. For three-dimensional (3-D) problems, the shapes used are tetrahedron and parallelepiped having straight or curved surfaces. Division of the domain into elements is called a **mesh**.

Step 3: In this step, over a typical element, a suitable approximation is chosen for the primary variable of the problem using **interpolation functions** or **shape functions**. You need to sole for the unknown values of the primary variable at some pre-selected points of the element, called as the **nodes**. Usually polynomials are chosen as the shape functions. The values of the primary variable at the nodes are called as the **degrees of freedom**.

Step 4: In this step, the approximation for the primary variable is substituted into the integral form. If the integral form is of variational type, the expression is minimized to get the algebraic equations for the unknown nodal values of the primary variable. If the integral form is of the weighted residual type, then it is set to zero to obtain the algebraic equations. In each case, the algebraic equations are obtained for each element called as the **elemental equations**. The elemental equations are then assembled over all the elements to obtain the algebraic equations for the whole domain called as the **global equations**.

Step 5: In this step, the algebraic equations are modified by incorporating the boundary conditions on the primary variable. The modified algebraic equations are solved to find the nodal values of the primary variable.

Step 6: Finally, the post-processing of the solution is done. That is, first the secondary variables of the problem are calculated from the solution. Then, the nodal values of the primary and secondary variables are used to construct their graphical variation over the domain.

Now that you have understood the basic steps involved in FEM. You have understood that solve a boundary value problem of any physical system, first you need to convert the differential equation of the problem into an integral form. This can be carried out either by variational technique or weighted residual technique. To transform the differential equation to an integral form by variational technique the need the basic understanding of calculus of variation. In fact, the calculus of variation is the foundation for the finite element method. Now, we will discuss the basics of calculus of variation.

2. Calculus of variation

2.1 Functional

To understand the calculus of variation, you must first understand the concept of a **functional**. Functional is an operator, which operates on a function and returns a number. In other words, functional is a function which is defined over a set of functions and whose range is a set of numbers. You can also define functional as a function of a function whose value depends on an independent function. A functional operator is represented by *I*.

As an illustration of a functional, consider a set of functions $u(x)$, which are functions of a single variable x for $0 \le x \le L$, which have the value zero at x=0 and possesses a continuous first derivative at all points of the interval [0, L]. Then you can define functional as

$$I_1 = \int_0^L u \mathrm{d}x \tag{4}$$

$$I_2 = \int_0^L \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x \tag{5}$$

$$I_3 = \int_0^L \alpha(x) u^2 \mathrm{d}x, \quad \alpha(x) \text{ is a known function} \tag{6}$$

$$I_4 = \int_0^L \beta(x) \log u \mathrm{d}x, \quad \beta(x) \text{ is a known function} \tag{7}$$

$$I_5 = c\, u\big|_{x=L/3}, \quad c \text{ is a known function} \tag{8}$$

In each of the above examples, the operator I_i takes a function $u(x)$ and returns a number either by integrating a quantity depending on u, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and some known functions or by evaluating u at some points of the interval. In general, a functional may be expressed as

$$I = \int_{x_1}^{x_2} F(x, u, u')\,\mathrm{d}x \tag{9}$$

Equation (9) does not include a type of functional defined by Eq. (8). The calculus of variation deals with the operations performed on functional. Thus, the branch of calculus, which deals with the operations performed on functional, is called as the **calculus of variation** or the **variational calculus**.

2.2 Variation and extremum of a functional — Euler-Lagrange equation

Let us consider the following functional:

$$I = \int_{x_1}^{x_2} F\left(x, u, u'\right)\mathrm{d}x. \tag{10}$$

We consider a function $u(x)$, which satisfies the essential boundary conditions: $u(x_1)=0$ and $u(x_2)=0$. The function is shown as a solid curve in Fig. 1.2. Now, in the neighborhood of $u(x)$, we consider another function $u(x)+\alpha\nu(x)$ shown as dotted line in Fig. 1.2, where ν is an arbitrary function of x and α is a very small arbitrary quantity. We assume that the function $u(x)+\alpha\nu(x)$ does not violate the essential boundary conditions. Thus,

$$\nu\left(x_1\right) = \nu\left(x_2\right) = 0. \tag{11}$$

The term αv is called the variation in u, represented by δu, where δ is known as the **variational operator**. Let us denote

$$U = u + \alpha v \tag{12}$$

Then,

$$U' = u' + \alpha v' \tag{13}$$

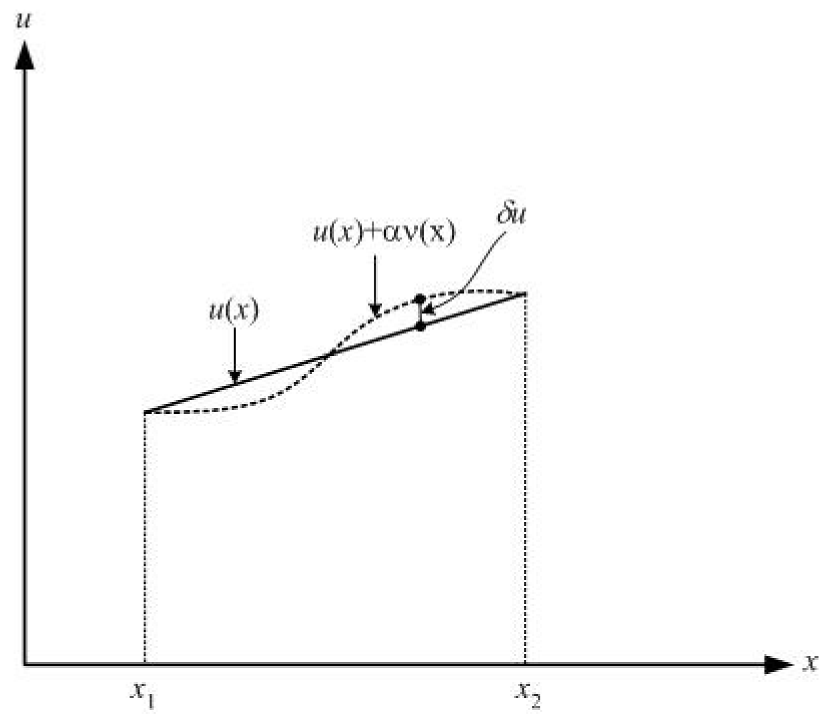

Fig. 1.2 Variation in *u*

Now, replacing u and u' in Eq. (10) by U and U', we get

$$I(\alpha) = \int_{x_1}^{x_2} F\left(x, u + \alpha v, u' + \alpha v'\right) \mathrm{d}x \tag{14}$$

By Taylor's theorem, we obtain

$$F\left(x, u + \alpha v, u' + \alpha v'\right) = F\left(x, u, u'\right) + \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' + \frac{\partial^2 F}{\partial u^2}\frac{(\alpha v)^2}{2!} + \frac{\partial^2 F}{\partial u \partial u'}\frac{\alpha^2 v v'}{2!} + \frac{\partial^2 F}{\partial u'^2}\frac{(\alpha v')^2}{2!} + \ldots\ldots\ldots \tag{15}$$

Substituting Eq. (15) in Eq. (14):

$$I = \int_{x_1}^{x_2} \left\{ F\left(x, u, u'\right) + \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' + \frac{\partial^2 F}{\partial u^2}\frac{(\alpha v)^2}{2!} + \frac{\partial^2 F}{\partial u \partial u'}\frac{\alpha^2 v v'}{2!} + \frac{\partial^2 F}{\partial u'^2}\frac{(\alpha v')^2}{2!} + \ldots\ldots\ldots \right\} \mathrm{d}x \tag{16}$$

The condition for extremization of the functional I is

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = 0. \tag{17}$$

Using Eq. (16) in Eq. (17) the extremization of the functional provides

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial u} v + \frac{\partial F}{\partial u'} v' \right\} \mathrm{d}x = 0. \tag{18}$$

The term $\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0}$ is called the first variation of the functional, denoted by δI. Thus, the condition for extremization of the functional can also be expressed as

$$\delta I = 0. \tag{19}$$

Integrating Eq. (18) by parts we obtain

$$\partial I = \int_{x_1}^{x_2} \frac{\partial F}{\partial u} v \mathrm{d}x + \left(\frac{\partial F}{\partial u'} v \right)\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'} \right) v \mathrm{d}x = 0,$$

$$\partial I = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'} \right) \right\} v \mathrm{d}x + \left(\frac{\partial F}{\partial u'} v \right)\Bigg|_{x_2} - \left(\frac{\partial F}{\partial u'} v \right)\Bigg|_{x_1} = 0. \tag{20}$$

Now, using Eq. (11) in Eq. (20):

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'} \right) \right\} v \mathrm{d}x = 0. \tag{21}$$

As v is arbitrary, from Eq. (21), we get

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'} \right) = 0. \tag{22}$$

Thus, extremization of the functional (Eq. 10) requires the satisfaction of the differential equation given by Eq. (22). This differential equation is known as the **Euler-Lagrange equation**. Now, substituting Eq. (22) in Eq. (20), we get

$$\left(\frac{\partial F}{\partial u'} v \right)\Bigg|_{x_2} - \left(\frac{\partial F}{\partial u'} v \right)\Bigg|_{x_1} = 0. \tag{23}$$

At the extreme points the value of v is zero or arbitrary. Thus, from Eq. (23), we can write:

At $x=x_1$ and x_2

$$\frac{\partial F}{\partial u'} v = 0. \tag{24}$$

Equation (24) suggests that at $x=x_1$ and x_2 either

$$v = 0 \ (i.e., u \text{ is prescribed at the boundaries}), \tag{25}$$

which is called the **essential boundary condition**,

or

$$\frac{\partial F}{\partial u'} = 0, \tag{26}$$

which is called the **natural boundary condition**.