ME301: Dynamics and Vibration of Machinery

Lecture 3

Two Degree of Freedom System

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In the last lecture, we started the analysis of 2 DOF system. We discussed about the different coordinate systems to express the vibratory motion of a system. Then we derived the equations of motion for a simple 2 DOF spring-mass system (Refer Fig. 3 of Lecture 2). So, we obtained the equations of motion for the spring-mass system in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{1}$$

One may use either generalized coordinate system or physical coordinate system to express the motion of the system. While using these coordinate systems, the mass and stiffness matrices may be coupled or uncoupled. When the mass matrix is coupled, the system is said to be **dynamically coupled**. When the system is such that its stiffness matrix is coupled, the system is said to be **statically coupled**. Similarly, when the mass matrix is uncoupled, the system is called as the **dynamically uncoupled** and when the stiffness matrix is uncoupled, the system is called as **statically uncoupled**. To understand the concept of coupling/uncoupling of mass and stiffness matrix, let us consider that the motion of a vibratory is represented by the following equation:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{2}$$

Now, if m_{12}=m_{21}=0, in Eq. (2), the mass matrix is said to be uncoupled. If any one of them is not zero, it is called coupled. Similarly, if k_{12}=k_{21}=0 in Eq. (2), in the stiffness matrix is said to be uncoupled. If any one of them is nonzero, then the stiffness matrix is said to be coupled. Consider that both the mass and stiffness matrix of Eq. (2) is uncoupled. Then, Eq. (2) can be written as

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3}$$

From Eq. (3), we can write the system equations as

$$m_{11}\ddot{x}_1 + k_{11}x_1 = 0, \tag{3a}$$

$$m_{22}\ddot{x}_2 + k_{22}x_2 = 0. \tag{3b}$$

You see that both the system equations (Eqs. 3a and 3b) are independent and individually they can be solved as that of a single degree of freedom system. The coordinate system for which both the mass and stiffness matrix are uncoupled is called as the **principal coordinate system**. Using the principal coordinate system, you can reduce a two DOF system into two equivalent single DOF systems. Thus, in Eq. (3), the coordinates x_1 and x_2 are the principal coordinates.

Now, if you go back to Eq. (1), you see that the mass matrix is uncoupled and the stiffness matrix is coupled. So, your spring-mass system (Fig. 3 of lecture 2) is dynamically uncoupled and statically coupled. At this stage, we consider some other examples of 2 DOF system and we will derive the equations of motion for them.

Example 3: Double pendulum

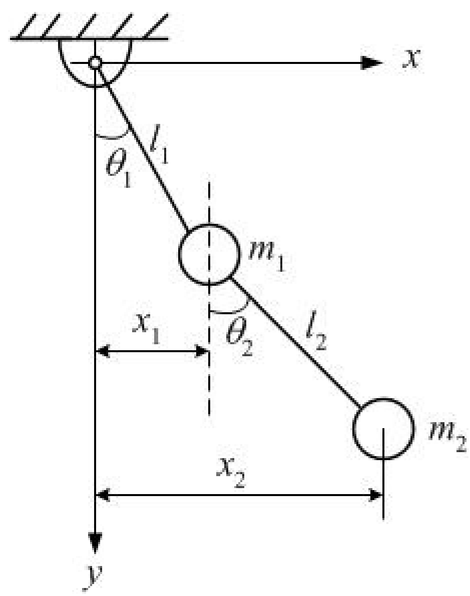
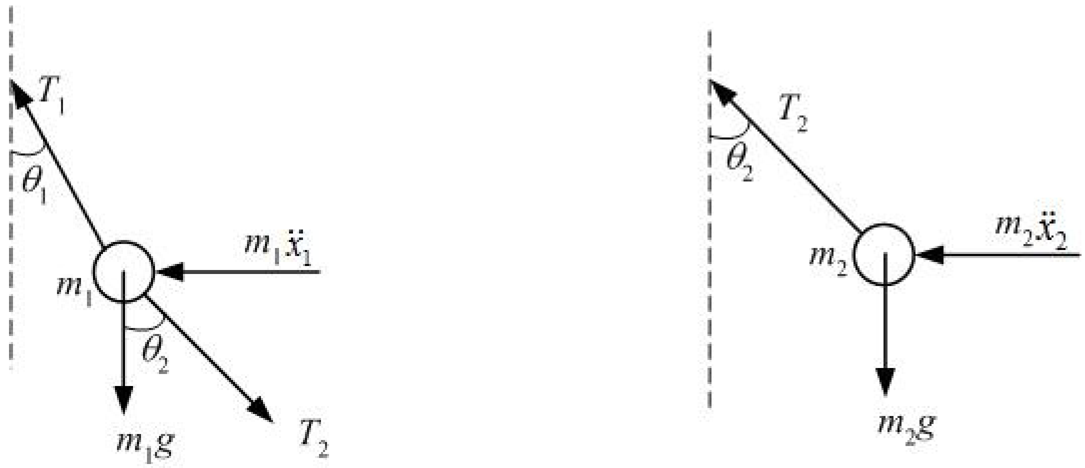


Fig. 1 Double pendulum

Let us first draw the free body diagrams for mass m_1 and m_2. Let, T_1 and T_2 be the tensions in the strings. Here, we are using the physical coordinates x_1 and x_2 to express the motion of the system.



(a) (b)

Fig. 2 FBDs of mass m_1 and m_2

Resolving the forces in the horizontal direction and using D'Alembert's principle we can write the equation of motion for mass m_1 as

$$m_1\ddot{x}_1 + T_1 \sin\theta_1 - T_2 \sin\theta_2 = 0. \tag{4}$$

From the configuration shown in Fig. 1,

$$\sin\theta_1 = \frac{x_1}{l_1}, \quad \sin\theta_2 = \frac{x_2 - x_1}{l_2}. \tag{5}$$

Using Eq. (5) in Eq. (4), we get

$$m_1\ddot{x}_1 + T_1\frac{x_1}{l_1} - T_2\frac{x_2 - x_1}{l_2} = 0,$$

$$m_1\ddot{x}_1 + \left(\frac{T_1}{l_1} + \frac{T_2}{l_2}\right)x_1 - \frac{T_2}{l_2}x_2 = 0. \tag{6}$$

Similarly, for mass m_2, we can the write the equation of motion as

$$m_2\ddot{x}_2 + T_2 \sin\theta_2 = 0,$$

$$m_2\ddot{x}_2 + T_2\frac{x_2 - x_1}{l_2} = 0,$$

$$m_2\ddot{x}_2 - \frac{T_2}{l_2}x_1 + \frac{T_2}{l_2}x_2 = 0. \tag{7}$$

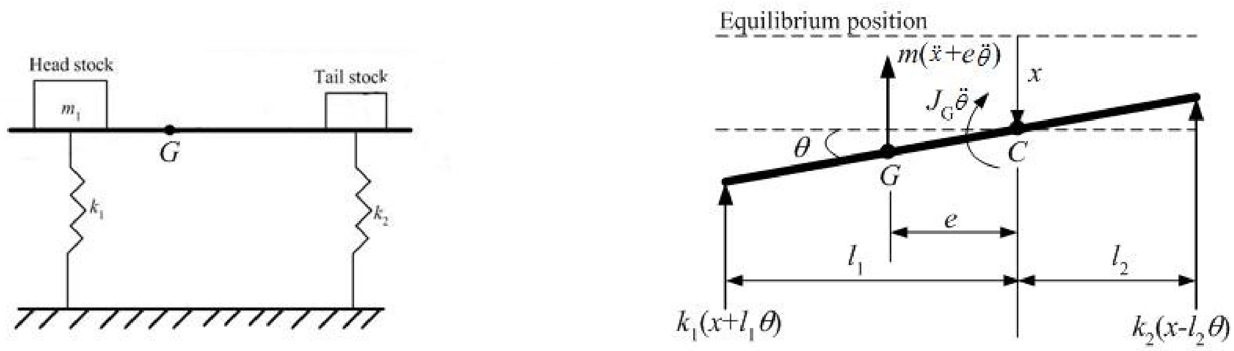
Writing Eqs. (6) and (7), in matrix form we obtain

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \left(\dfrac{T_1}{l_1} + \dfrac{T_2}{l_2} \right) & -\dfrac{T_2}{l_2} \\ -\dfrac{T_2}{l_2} & \dfrac{T_2}{l_2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{8}$$

Eq. (8) suggests that the present system is dynamically uncoupled and statically coupled.

Example 3: Lathe machine

Now, we take the example of the lathe machine which can be modeled as a two degree of freedom system. The configuration of the lathe machine as a 2 DOF system has already been discussed in lecture 2. The machine can be modeled as a rigid bar with its centre of mass not coinciding with its geometric centre and supported by two springs k_1 and k_2. Here, in expressing the motion of the lathe machine we consider the angular displacement and its transverse displacement from mean position as shown in Fig. 3.



(a) (b)

Fig. 3 Lathe machine as a 2 DOF system

Fig. 3 (b) shows the FBD of the system. The point G is the centre of mass and point C represents a point on the bar at which the coordinates of the system is defined. The point C is at a distance of l_1 from the left end and it is at a distance of l_2 from the right end. The distance between the points G and C is e. Let us assume that the linear displacement of point C is x from the mean position and θ is the angular displacement about point C. Now, the equation of motion of this system can be obtained by using D'Alembert's principle. The force balance provides

$$m\left(\ddot{x} + e\ddot{\theta}\right) + k_1\left(x + l_1\theta\right) + k_2\left(x - l_2\theta\right) = 0,$$

$$m\left(\ddot{x} + e\ddot{\theta}\right) + \left(k_1 + k_2\right)x + \left(k_1 l_1 - k_2 l_2\right)\theta = 0. \tag{9}$$

Taking moment of all the forces about point C, we get

$$J_G\ddot{\theta}+m\left(\ddot{x}+e\ddot{\theta}\right)e+k_1\left(x+l_1\theta\right)l_1-k_2\left(x-l_2\theta\right)l_2=0. \tag{10}$$

Since $J_G+me^2=J_C$, Eq. (10) can be rewritten as

$$me\ddot{x}+J_C\ddot{\theta}+\left(k_1l_1-k_2l_2\right)x+\left(k_1{l_1}^2+k_2{l_2}^2\right)\theta=0. \tag{11}$$

Eqs. (9) and (11) can be written in matrix form as

$$\begin{bmatrix} m & me \\ me & J_C \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & k_1l_1-k_2l_2 \\ k_1l_1-k_2l_2 & k_1{l_1}^2+k_2{l_2}^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{12}$$

Eq. (12) suggests that the system is both dynamically and statically coupled. Now, depending on the position of point C, the following cases are considered:

Case 1: The point G and point C coincides, *i.e.*, e=0. In this case, the system equation (Eq. 12) reduces to

$$\begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & k_1l_1-k_2l_2 \\ k_1l_1-k_2l_2 & k_1{l_1}^2+k_2{l_2}^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{13}$$

Thus, in this case, the system is statically coupled, but dynamically uncoupled.

Case 2: If point G and point C coincides and $k_1l_1=k_2l_2$, the system becomes both dynamically and statically uncoupled and in that case, we obtain uncoupled x and θ vibrations. For this condition, the coordinates x and θ will be known as the principal coordinates. The equation of motion reduces to

$$\begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & 0 \\ 0 & k_1{l_1}^2+k_2{l_2}^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{14}$$

Case 3: If $k_1l_1=k_2l_2$, the system equation reduces to

$$\begin{bmatrix} m & me \\ me & J_C \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & 0 \\ 0 & k_1{l_1}^2+k_2{l_2}^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{15}$$

Thus, in this case, the system is dynamically coupled and statically uncoupled.