ME301: Dynamics and Vibration of Machinery

Lecture 2

Two Degree of Freedom System

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You have already studied the single degree of freedom (DOF) system. You dealt with free and force vibration in single degree of freedom system. In case of free vibration of single DOF system, you have studied the system without damping and the system with damping. In case of system with damping, you have studied three different cases— underdamped case, critical damping case and overdamped case. You also studied the different vibration measuring instruments.

Today we are going to study two DOF systems. Many machine components cannot be modeled as a single DOF system. So, they may be modeled as a two DOF system. In our previous class, we discussed that for two DOF system, we need two independent coordinates to describe the system dynamics. For example, the oscillatory motion of a double pendulum or two masses connected in series by springs are the simplest forms of two DOF systems. Let us consider the example of a lathe machine. The lathe machine with its head stock and tail stock can be modeled as a two DOF system. The system is shown in Fig. 1.

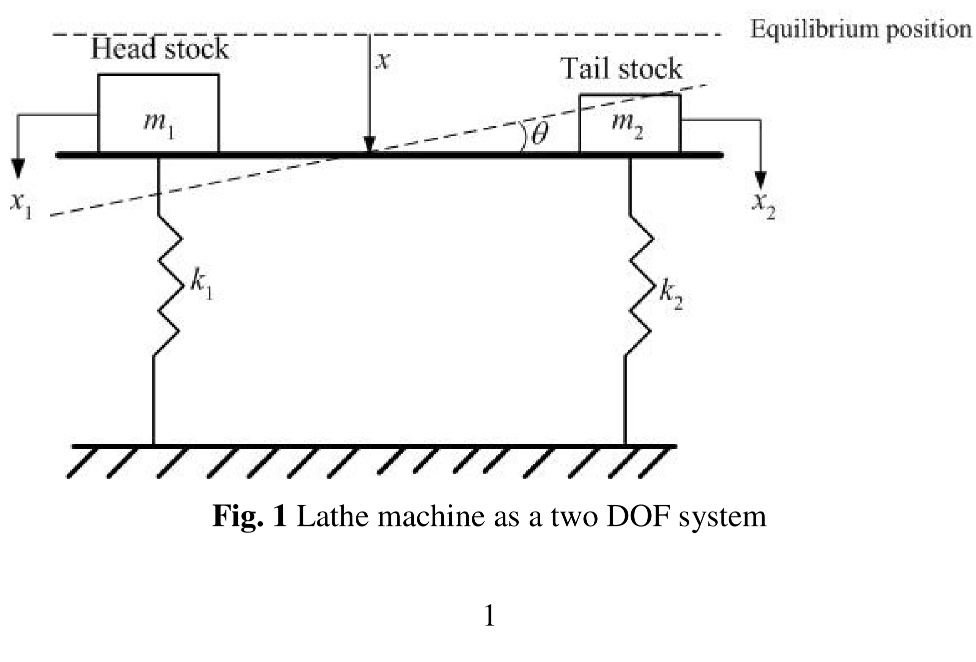

Fig. 1 Lathe machine as a two DOF system

You can express the vibration or oscillatory motion of the lathe machine in different ways. You can express the motion of masses m_1 and m_2 by the transverse displacements x_1 and x_2 as shown in Fig. 1. Also, you may express the vibration of the motion in terms of the angular displacement θ and its transverse displacement x from the equilibrium position as shown in Fig.1 (the configuration is shown as dotted lines).

Coordinate system for expressing the motion of a vibratory system

For expressing the motion of a vibratory system, you may use different coordinate systems. So, understand the different coordinate systems used in the motion of a vibratory system, let us consider the example of a double pendulum as shown in Fig. 2. The angular displacement of mass m_1 is θ_1 and the angular displacement of mass m_2 is θ_2 from their mean positions. These coordinates θ_1 and θ_2 of masses m_1 and m_2, which are displacements from the mean positions, are called as the **generalized coordinate system**. The generalized coordinate systems are the minimum number of coordinates which are needed to express the vibratory motion of the system.

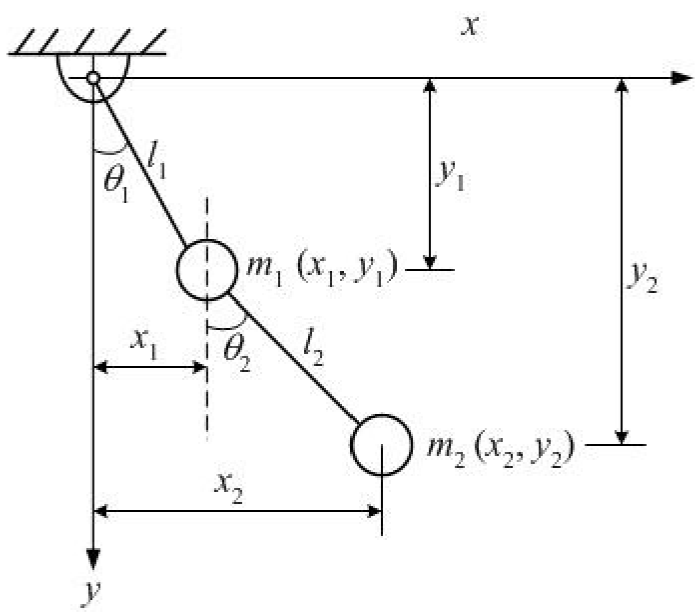

Fig. 2 Representation of motion of a double pendulum with physical and generalized coordinate systems

You can also express the motion of the masses in Fig. 2 by fixing fixed reference axes *x*-*y*. In that case, the motion of m_1 can be expressed by the coordinate (x_1, y_1) and the moton of mass m_2 can be expressed by the coordinate (x_2, y_2). These coordinate with reference to a fixed coordinate axes attached to the system are called as **physical coordinate system**. So, you can express the motion of a vibratory system either by using the generalized coordinate or physical coordinate system. Let us now derive the equations of motion for some 2 DOF vibratory systems.

Equations of motion for 2 DOF system

Example 1: Consider the following spring-mass system.

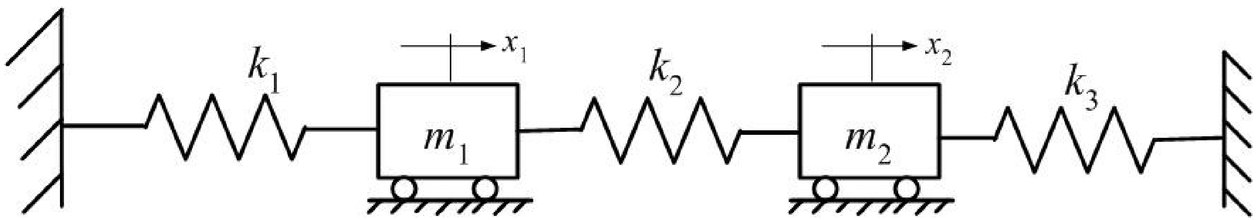

Fig. 3 Spring-mass system (2 DOF)

This is a two DOF system. You can write the equations of motion for this system using either **Newton's second law** of motion or **D'Alembert's principle**. Here, in deriving the equations of motion, we use the generalized coordinates x_1 and x_2 for masses m_1 and m_2, respectively. First draw the free body diagram of each mass.

For mass m_1:

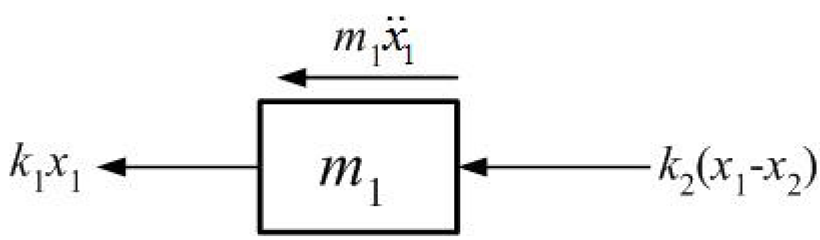

Fig. 3a FBD of mass m_1

Using D'Alembert's principle for mass m_1, we can write the following equation of motion from the free body diagram (FBD) (Fig. 3a)

$$m_1\ddot{x}_1 + k_1x_1 + k_2\left(x_1 - x_2\right) = 0,$$

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0. \tag{1}$$

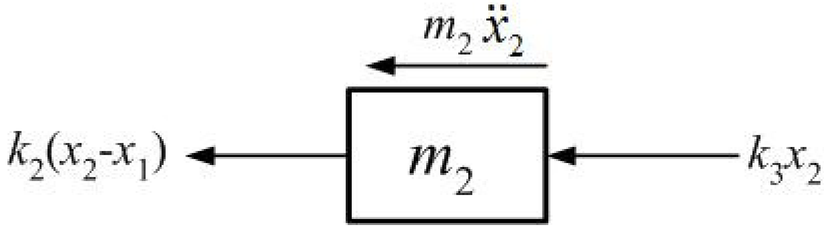

Fig. 3b FBD of mass m_2

Similarly, for mass m_2, we get (refer to Fig. 3b)

$$m_2\ddot{x}_2 + k_2\left(x_2 - x_1\right) + k_3 x_2 = 0,$$

$$m_2\ddot{x}_2 + \left(k_2 + k_3\right)x_1 - k_2 x_1 = 0. \tag{2}$$

Important points:

(i) Inertia force acts opposite to the direction of acceleration. In FBDs shown in Figs. 3a and 3b, the direction of acceleration is towards right and thus, the direction of inertia forces are shown towards left.

(ii) For spring k_2, you can assume either $x_2 > x_1$ or $x_1 > x_2$. First assume that $x_2 > x_1$. The spring with stiffness k_2 will pull the mass towards right by a force equal to $k_2(x_2\text{-}x_1)$ and it is extended by $(x_2\text{-}x_1)$ towards right. The force exerted by the spring k_2 on mass m_1 is shown as $k_2(x_1\text{-}x_2)$ towards left in Fig. 3a. The spring will exert a force of $k_2(x_2\text{-}x_1)$ on mass m_2 towards left as shown in Fig. 3b. Similarly when you assume $x_1 > x_2$, the spring gets compressed by an amount $(x_2\text{-}x_1)$. You can show the spring force exerted by spring k_2 on the masses in the FBDs. In both the cases, you will find that the FBDs remain unchanged.

Now, let us write the equations of motion (Eqs. 1 and 2) in matrix form. We get

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3}$$

In Eq. (3), the matrix, $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ is called as the mass matrix and the matrix, $\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$ is called as the stiffness matrix.