TUTORIAL (UNIT I)

MS 105: Mathematics II

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1. For the following exercises, prove that T is a linear map, find the bases for both $N(T)$ and $R(T)$, compute the nullity and rank of T and verify the rank-nullity theorem.

   (1) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.
   (2) $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.
   (3) $T : M_{2\times3}(F) \to M_{2\times2}(F)$ such that $T\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}$
   (4) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ such that $T(f(x)) = xf(x) + f'(x)$.

2. Determine the matrix associated with the following linear transformations in terms of the standard bases.

   (1) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.
   (2) $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.
   (3) $T : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ such that $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + b) + (2d)x + bx^2$.
   (4) $T : P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ such that $T(f(x)) = \begin{bmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{bmatrix}$.

3. Find the characteristic equations of the following matrices and hence find the eigenvalues and eigenvectors of the matrices

   (a) $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

   (e) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$

4. Show that the matrices A and $P^{-1}AP$ have the same eigenvalues.

5. Let A and B are square matrices of same order. Then show that AB and BA have the same eigenvalues but different eigenvectors. Further prove that AB^{-1} and $B^{-1}A$ have the same eigenvalues but different eigenvectors.

6. Verify Cayley-Hamilton theorem for the following matrices

   (a) $\begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 7 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

7. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. Find A^{-1}. Find A^8.

8. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Find A^{-1}. Find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

9. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Find A^{-1}. Find $A^8 - 5A^7 + 7A^6 - 3A^5 - 5A^3 + 8A^2 - 2A + I$.