TUTORIAL

MS 105: Mathematics II

Unit-2 (Complex Analysis)

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(1) Evaluate the following limits using $\epsilon - \delta$ definition

(a) $\lim\limits_{z\to 2i} \frac{z^2+4}{z-2i}$ (b) $\lim\limits_{z\to 1} \frac{iz}{3}$ (c) $\lim\limits_{z\to 0} \text{Im}\left(\frac{z}{|z|+1}\right)$ (d) $\lim\limits_{z\to i}(z^2+2z)$ (e) $\lim\limits_{z\to 1+i}(2+i)z$

(2) Compute the following limits

(a) $\lim\limits_{z\to -4i} \frac{z^2+16}{z+4i}$ (b) $\lim\limits_{z\to i} \frac{z^6+1}{z^2+1}$ (c) $\lim\limits_{z\to 0} \frac{1-\cos z}{z^2}$ (d) $\lim\limits_{z\to -i} \frac{iz^3+1}{z^2+1}$

(e) $\lim\limits_{z\to 0} ze^z$ (f) $\lim\limits_{z\to 0} \frac{3\bar{z}^2}{z}$ (g) $\lim\limits_{z\to\infty}[\sqrt{z-2i}-\sqrt{z-i}]$ (h) $\lim\limits_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i}$

(3) Show that the following limits do not exist.

(a) $\lim\limits_{z\to 0}(\frac{z}{\bar{z}})^2$ (b) $\lim\limits_{z\to 0}\frac{\text{Re}(z)}{|z|}$ (c) $\lim\limits_{z\to 0}\frac{z}{\text{Re}(z)}$ (d) $\lim\limits_{z\to 0}\frac{\text{Im}(z)}{z}$

(e) $\lim\limits_{z\to 0}\frac{\text{Im}(z^2)}{|z|^2}$ (f) $\lim\limits_{z\to 0}\frac{z^2}{|z|^2}$ (g) $\lim\limits_{z\to 0}\frac{\text{Re}(z)+\text{Im}(z)}{|z|^2}$

(4) Test the following functions for continuity

(a) $f(z) = \begin{cases} \frac{z^2+9}{z-3i}, & z \neq 3i \\ 4+6i, & z = 3i \end{cases}$ at $z = 3i$

(b) $f(z) = \begin{cases} \bar{z}^3, & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z = 0$

(c) $f(z) = \begin{cases} \text{Im}(z), & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z = 0$

(d) $f(z) = \begin{cases} \frac{\text{Re}(z)\text{Im}(z)}{z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z = 0$

(e) $f(z) = \frac{\text{Re}(z)}{1+|z|}$ at $z = 0$

(f) $f(z) = \begin{cases} \frac{\text{Re}(z^2)}{|z|^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z = 0$

(g) $f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z = 0$

(5) Find the points of discontinuity for the following functions

(a) $f(z) = \frac{2z-3}{z^2+2z+2}$ (b) $f(z) = \frac{3z^2+4}{z^4-16}$ (c) $f(z) = \frac{z}{z^4+1}$ (d) $f(z) = \frac{1}{z} - \sec z$

(6) Find $f(1-i)$ if $f(z) = \frac{z^2-2z+2}{z^2+2i}$ is continuoues at $z = 1-i$.

(7) Show that $f(z) = \frac{1}{z}$ is continuous in any domain not containing the origin.

(8) Show that $f(z) = z^2\bar{z}$ is not differentiable anywhere in the argand plane.

(9) Prove that $f(z) = |\bar{z} - a|^2$ is differentiable only at $z = a$.

(10) Find the points at which $f(z) = z\text{Im}(z)$ is differentiable.

(11) Show that the complex valued function defined by $f(z) = \begin{cases} \frac{\bar{z}^3}{z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is continuous everywhere on the complex plane but is not differentiable at the origin.

(12) Find the values of the constants a, b, c for which $f(x+iy) = ax + by + i(cx + y)$ is analytic on $\mathbb{C}$.

(13) Show that $f(z) = \sin(\bar{z})$ is nowhere analytic on $\mathbb{C}$.

(14) Show that $f(z) = \sqrt{|\text{Re}(z)\text{Im}(z)|}$ is not diiferentiable at the origin even though Cauchy-Riemann equations are satisfied there.

(15) Show that the function defined by $f(x+iy) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & x = y = 0 \end{cases}$ is continuous and that Cauchy-Riemann equations hold at the origin but yet is not differentiable there.

(16) Show that the complex valued function defined by $f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is continuous and that Cauchy-Riemann equations hold at the origin but yet is not differentiable there.