Problem Set I

MS 105/ MS 103: Mathematics II

The fields $\mathbb{F}$ considered in this Problem Set is either $\mathbb{R}$ or $\mathbb{C}$.

1. Find the Echelon form of the following matrices and hence find the rank of the matrices.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -3 & 4 & 9 \\ 1 & 0 & -1 & 1 & 1 \\ 3 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 2 & 9 \\ 3 & 1 & 0 & 3 & 9 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ (h) $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ (j) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

2. Determine consistency of the following systems of linear equations

(1) $5x + 3y + 7z = 4, \quad 3x + 26y + 2z = 9, \quad 7x + 2y + 10z = 5$

(2) $x_1 + x_2 + 2x_3 + x_4 = 5, \quad 2x_1 + 3x_2 - x_3 - 2x_4 = 2, \quad 4x_1 + 5x_2 + 3x_3 = 7$

(3) $3x_1 + 2x_2 + x_3 = 3, \quad 2x_1 + x_2 + x_3 = 0, \quad 6x_1 + 2x_2 + 4x_3 = 6$

(4) $2x_1 + x_2 + 5x_3 + x_4 = 5, \quad x_1 + x_2 - 3x_3 - 4x_4 = -1, \quad 3x_1 + 6x_2 - 2x_3 + x_4 = 8, \quad 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$

(5) $x_1+x_2-2x_3+x_4+3x_5 = 1, \quad 2x_1-x_2+2x_3+2x_4+6x_5 = 2, \quad 3x_1+2x_2-4x_3-3x_4-9x_5 = 3$

3. Find the values of a and b for which the system has (i) no solution (ii) unique solution (iii) infinitely many solution for

(a) $2x + 3y + 5z = 9, \quad 7x + 3y - 2z = 8, \quad 2x + 3y + az = b$

(b) $x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + az = b$

4. Determine b such that the system of homogeneous equation $2x + y + 2z = 0, \quad x + y + 3z = 0, \quad x + 3y + bz = 0$ has (i) trivial solution (ii) non-trivial solution.

5. Determine the value of b for which the systems of equations have non-trivial solutions.

(a) $(b-1)x+(4b-2)y+(b+3)z = 0, \quad (b-1)x+(3b+1)y+2bz = 0, \quad 2x+(3b+1)y+3(b-1)z = 0$

(b) $2x + 3by + (3b+4)z = 0, \ x + (b+4)y + (4b+2)z = 0, \ x + 2(b+1)y + (3b+4)z = 0.$

6. Solve the following systems of linear equations by Gaussian elimination method.

(a) $2x_1+2x_2+x_3+2x_4 = 7, \ -x_1+2x_2+x_4 = -2, \ -3x_1+x_2+2x_3+x_4 = -3, \ -x_1+2x_4 = 0$

(b) $2x_1 + 5x_2 + 2x_3 - 3x_4 = 3, \quad 3x_1 + 6x_2 + 5x_3 + 2x_4 = 2, \quad 4x_1 + 5x_2 + 14x_3 + 14x_4 = 11, \ 5x_1 + 10x_2 + 8x_3 + 4x_4 = 4$

(c) $x_1 + 2x_2 - x_3 = 3, \ 3x_1 - x_2 + 2x_3 = 1, \ 2x_1 - 2x_2 + 3x_3 = 2, \ x_1 - x_2 + x_3 = -1$

(d) $2x_1 + x_2 + 3x_3 = 1, \ 4x_1 + 4x_2 + 7x_3 = 1, \ 2x_1 + 5x_2 + 9x_3 = 3$

(e) $10x_1 - 7x_2 + 3x_3 + 5x_4 = 6, \quad -6x_1 + 8x_2 - x_3 - 4x_4 = 5, \quad 3x_1 + x_2 + 4x_3 + 11x_4 = 2, \ 5x_1 - 9x_2 - 2x_3 + 4x_4 = 7.$

7. Solve the following systems of linear equations by Gauss-Jordan method
   (a) $2x_1 + x_2 + 4x_3 = 12,\ \ 8x_1 - 3x_2 + 2x_3 = 20,\ \ 4x_1 + 11x_2 - x_3 = 33$
   (b) $x_1 + 4x_2 - x_3 = -5,\ \ x_1 + x_2 - 6x_3 = -12,\ \ 3x_1 - x_2 - x_3 = 4$
   (c) $x_1 - 2x_2 + x_3 + 2x_4 = 1,\ \ x_1 + x_2 - x_3 + x_4 = 2,\ \ x_1 + 7x_2 - 5x_3 - x_4 = 4$
   (d) $-x_1 + x_2 + x_3 + x_4 = 1,\ \ x_1 - x_2 + x_3 + x_4 = 0,\ \ x_1 + x_2 - x_3 + x_4 = 0,\ \ x_1 + x_2 + x_3 - x_4 = 0$
   (e) $x_1 + x_2 - 2x_3 + 3x_4 = 0,\ \ x_1 - 2x_2 + x_3 - x_4 = 0,\ \ 4x_1 + x_2 - 5x_3 + 8x_4 = 0,\ \ 5x_1 - 7x_2 + 2x_3 - x_4 = 0$

8. Solve the following systems of linear equations by LU decomposition method
   (a) $x - y + z = 2,\ \ 2x + 3y - z = 5,\ \ x + y - z = 0$
   (b) $2x - z = 1,\ \ 5x + y = 7,\ \ y + 3z = 5$
   (c) $x_1 - 2x_2 + x_3 + 2x_4 = 1,\ \ x_1 + x_2 - x_3 + x_4 = 2,\ \ x_1 + 7x_2 - 5x_3 - x_4 = 4$
   (d) $A = \begin{bmatrix} 9 & 3 & 3 & 3 \\ 3 & 10 & -2 & -2 \\ 3 & -2 & 18 & 10 \\ 3 & -2 & 10 & 10 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad B = \begin{bmatrix} 24 \\ 17 \\ 45 \\ 29 \end{bmatrix}$

9. Prove that the set of all $m \times n$ matrices with entries from a field $\mathbb{F}$, denoted by $M_{m\times n}$ with the following operations of matrix addition and scaler multiplication: for $A,\ B \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$,
$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} \text{ and } (cA)_{ij} = c(A)_{ij}.$$
is a vector space.

10. Let S be any non-empty set and $\mathbb{F}$ be any field. Prove that the set of all functions from S to $\mathbb{F}$, denoted by $\mathcal{F}(S, \mathbb{F})$ is a vector space with the following operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}(S, \mathbb{F})$ and $c \in \mathbb{F}$ by
$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = cf(s)$$
for each $s \in S$.

11. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{F}\}$, where $\mathbb{F}$ is a field. Define addition of elements of V coordinatewise, and for $c \in \mathbb{F}$ and $(a_1, a_2) \in V$, define
$$c(a_1, a_2) = (a_1, 0).$$
Is V a vector space over $\mathbb{F}$ with these operations? Justify your answer.

12. Let V and W be vector spaces over a field $\mathbb{F}$. Let
$$Z = \{(v, w) : v \in V \text{ and } w \in W\}.$$
Prove that Z is a vector space over $\mathbb{F}$ with the operations
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

13. For each of the following list of vectors, determine whether the first vector can be expressed as a linear combination of the other two:
   (a) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$ in $\mathbb{R}^3$
   (b) $(3, 4, 1), (1, -2, 1), (-2, -1, 1)$ in $\mathbb{R}^3$
   (c) $(5, 1, -5), (1, -2, -3), (-2, 3, -4)$ in $\mathbb{R}^3$
   (d) $x^3 - 3x + 5, x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1$ in $P_3(\mathbb{R})$
   (e) $6x^3 - 3x^2 + x + 2, x^3 - x^2 + 2x + 3, 2x^3 - 3x + 1$ in $P_3(\mathbb{R})$

14. Determine whether the given vectors is in the span of S:

(a) $(-1,1,1,2)$, $S=\{(1,0,1,-1),(0,1,1,1)\}$
(b) $-x^3+2x^2+3x+3$, $S=\{x^3+x^2+x+1, x^2+x+1, x+1\}$
(c) $\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}$, $S=\left\{\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\right\}$
(d) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $S=\left\{\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\right\}$

15. Show that if $M_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $M_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $M_3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ then the span of $\{M_1,M_2,M_3\}$ is the set of all symmetric 2×2 matrices.

16. Show that the matrices $M_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $M_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $M_3=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ and $M_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ generate $M_{2\times 2}(\mathbb{F})$.

17. Show that the polynomials x^2+3x-2, $2x^2+5x-3$ and $-x^2-4x+4$ generate $P_2(\mathbb{R})$.

18. Show that in $M_{2\times 3}(\mathbb{R})$, the set $\left\{\begin{bmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{bmatrix}, \begin{bmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{bmatrix}, \begin{bmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{bmatrix}\right\}$ is linearly dependent.

19. Prove that the set $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ is linearly independent.

20. Determine whether the following sets are linearly dependent or linearly independent

(a) $\{x^3+2x^2, -x^2+3x+1, x^3-x^2+2x-1\}$ in $P_3(\mathbb{R})$
(b) $\{(1,-1,2),(1,-2,1),(1,1,4)\}$ in $\mathbb{R}^3$
(c) $\left\{\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 2\\ 1 & 0,\end{bmatrix}, \begin{bmatrix}2 & 1\\ -4 & 4\end{bmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$
(d) $\{x^4-x^3+5x^2-8x+6, -x^4+x^3-5x^2+5x-3, x^4+3x^2-3x+5, 2x^4+x^3+4x^2+8x\}$ in $P_4(\mathbb{R})$

21. Let V be a vector space and let $S_1\subseteq S_2\subseteq V$. If S_1 is linearly dependent then prove that S_2 is also linearly dependent.

22. Let V be a vector space and let $S_1\subseteq S_2\subseteq V$. If S_2 is linearly independent then prove that S_1 is also linearly independent.

23. In $M_{m\times n}(\mathbb{F})$, let E_{ij} denote the matrix whose only non-zero entry is a 1 in the i-th row and j-th column. Then prove that $\{E_{ij}: 1\le i\le m, 1\le j\le n\}$ is a basis for $M_{m\times n}(\mathbb{F})$.

24. Prove that $\{x^2+3x-2, 2x^2+5x-3, -x^2-4x+4\}$ is a basis for $P_2(\mathbb{R})$.

25. Determine whether the following sets are basis for the given vector spaces, justify your answer.

(a) $\{(1,2,-1),(1,0,2),(2,1,1)\}$ in $\mathbb{R}^3(\mathbb{R})$
(b) $\{(1,-3,-2),(-3,1,3),(-2,-10,-2)\}$ in $\mathbb{R}^3(\mathbb{R})$
(c) $\{1+2x-x^2, 4-2x+x^2, -1+18x-9x^2\}$ in $P_2(\mathbb{R})$
(d) $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$ in $P_2(\mathbb{R})$

26. Prove that the followings are linear transformations.
 (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1, -a_2, 2a_3)$
 (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$
 (c) $T : M_{2\times3}(\mathbb{F}) \to M_{2\times2}(\mathbb{F})$ defined by

$$T \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \to \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}$$

 (d) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$
 (e) $T : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ defined by $T(A) = trace(A)$

27. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is not linear transformation.
 (a) $T(a_1, a_2) = (1, a_2)$
 (b) $T(a_1, a_2) = (a_1, a_2^2)$
 (c) $T(a_1, a_2) = (sina_1, 0)$
 (d) $T(a_1, a_2) = (|a_1|, a_2)$
 (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

28. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. If $T(1, 0) = (1, 4)$ and $T(1, 1) = (2, 5)$ then what is $T(2, 3)$?

29. Let $V = C(\mathbb{R})$, the vector space of continuous real-valued functions on $\mathbb{R}$. Let $a, b \in \mathbb{R}$, $a \leq b$. Define $T : V \to R$ by $T(f) = \int_a^b f(t)dt$ for all $f \in V$. Then prove that T is a linear transformation.

30. Find the range and kernel of the linear transformations in Question 26.

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Hints to some problems

4. The system $Ax = b$ has
 (1) No solution if Rank$(A) \neq$ Rank$((A|b))$
 (2) Unique solution if Rank$(A) =$ Rank$((A|b)) =$ number of unknowns
 (3) Infinite solutions if Rank$(A) =$ Rank$((A|b)) <$ number of unknowns

 The reduced echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 0 & \frac{-15}{2} & \frac{-39}{2} & \frac{-47}{2} \\ 0 & 0 & a - \frac{15}{2} & b - \frac{27}{2} \end{bmatrix}$$

 Hence the system has
 (1) No solution if $a = \frac{15}{2}$, $b \neq \frac{27}{2}$
 (2) Unique solution if $a \neq \frac{15}{2}$
 (3) Infinite solutions if $a = \frac{15}{2}$, $b = \frac{27}{2}$s

28. $(1, 0)$ and $(1, 1)$ are linearly independent (check). Now $(2, 3) = -(1, 0) + 3(1, 1)$. Since T is linear, we have $T(2, 3) = -T((1, 0)) + 3T((1, 1)) = (5, 11)$.

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