Class Note & Practice Problems, 2020 (Topic: Two person zero-sum game)
MI504/MS510 : Mathematical Programming

[***All symbols used in this paper have their usual meaning unless otherwise stated***]

1. For each of the following payoff matrices find v^+ and v^-. Determine the saddle points, if exist?.
(a) $\begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 6 & 5 \\ 2 & 0 & 2 \\ 4 & 6 & 4 \end{bmatrix}$
2. Find a necessary and sufficient condition that the payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has saddle points?
3. Find a necessary and sufficient condition that the payoff matrix $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ has saddle points?
4. Determine whether there are saddle points, using dominance rule.
(a) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 3 & 3 & 1 \\ 2 & 2 & 0 & 1 \end{bmatrix}$
5. In the game given by the payoff matrix $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ find the expected payoff $E(X, Y)$ to player A when A uses the mixed strategy $A = (\frac{1}{3}, \frac{2}{3})$ and B uses the mixed strategy $Y = (\frac{3}{4}, \frac{1}{4})$.
6. If $X = (.2, .8)$ and $Y = (.5, 0, .5)$ are mixed strategies for A and B in the game given by the payoff matrix $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$, then compute the expected payoff $E(X, Y)$ for A.
7. If (X_1, Y_1) and (X_2, Y_2) are equilibrium pairs of a two person zero-sum game, then prove that (X_2, Y_1) is an equilibrium pair.
8. Let $X = (.6, .3, .1)$ and $Y = (.3, .4, ..3)$ be mixed strategies for players A and B in the two person zero-sum game given by the payoff matrix $\begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$. Determine whether (X, Y) is an equilibrium pair.

Mixed strategy:

Consider the following payoff matrix $\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$.

Here $v^- = 0$ and $v^+ = 1$. There is no saddle point.

(1) If player A consistently plays either of his two strategies then player B can choose a strategy that will give A at most $v^- = 0$. (OR)

(II) If player B consistently plays one of his strategies, then player A can choose a strategy that will ensure A of a gain of at least $v^+ = 1$.

* A and B vary their choice randomly so that their opposition can not guess how they will play; ~~so~~ for better payoffs.

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A has m strategies $A_1, A_2, \dots A_m$.
B has n strategies $B_1, B_2, \dots B_n$.

⊛ A mixed strategy for A is an $m$tuple $(x_1, x_2, \dots x_m)$ of non negative real numbers such that $x_1 + x_2 + \dots + x_n = 1$

The component x_i represent the probability that strategy A_i or row i will be used by player A.

So $x_i = \text{Prob}$ (A uses strategy A_i)

⊛ A mixed strategy for B is a $1 \times n$ vector $(y_1, y_2, \dots y_n)$ ~~where~~ such that $y_i \geq 0$ & $\sum y_i = 1$.

$y_i = \text{Prob}$ (B uses strategy B_i)

$A = (a_{ij})$ be the payoff matrix

Defn: Given a choice of mixed stategy $X = (x_1 x_2 \cdots x_m)$ for player A and $Y = (y_1 y_2 \cdots y_n)$ for player B choosen independenly. the expected payoff to player A of the game is —

$$E(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \times \text{Prob}(\text{Player A uses } i \text{ and B uses } j)$$

$$= \sum \sum a_{ij} \ \text{Prob}(\text{Player A uses } i) \times \text{Prob}(\text{Player B uses } j)$$

$$= \sum \sum a_{ij} \, x_i y_j = \boxed{X A Y^T}$$

$P(A \cap B) = P(A)P(B)$ when A B are independent event

ex: $X = (.2 \ \ .8)$ $Y = (.5, 0, .5)$. are mixed stategies for A and B of the game given by —
$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix}$. compute $E(X,Y)$.

Equilibriumpair:

A pair of mixed stategies (X^*, Y^*) for players A and B, is called equilibriumpair if

(I) $E(X^*, Y^*) \leq E(X^* Y)$ for any mixed stategy Y for B

(II) $E(X, Y^*) \leq E(X^* Y^*)$ for any mixed stategy X for A.

Note: Every pure stategy can be thought of as a mixed stategy.

for example: pure stategy A_1 can be represented as the mixed stategy $(1, 0, 0 \cdots 0)$.

$A_2 \rightarrow (0 1, 0 \ 0 \cdots 0)$

Th^m: 1 If $(X_1 Y_1)$ and $(X_2 Y_2)$ are equilibrium pairs of a game, then $E(X_1 Y_1) = E(X_2 Y_2)$.

Proof: $E(X_1 Y_1) \le E(X_1, Y_2) \le E(X_2, Y_2) \longrightarrow (1)$

$E(X_2 Y_2) \le E(X_2, Y_1) \le E(X_1 Y_1) \longrightarrow (11)$

* Pay offs for all equilibrium pairs of a game are equal. That common value is called the value of the game.

Von Neumann's Th^m: Every game has atleast one equilibrium pair of mixed strategies.

By this th^m: every game has a value.

Th^m: 2 If $(X_1 Y_1)$ and $(X_2 Y_2)$ are equilibrium pairs of a game, then so is (X_1, Y_2).

Pf: $E(X_1 Y_1) = E(X_2 Y_2)$.

(1) & (11) $\Rightarrow E(X_1 Y_2) = E(X_1 Y_1)$.

So $E(X_1 Y_2) = E(X_1 Y_1) \le E(X_1 Y)$ for all Y

Also $E(X_1 Y_2) = E(X_2 Y_2) \ge E(X, Y_2)$ for all X.

Thus $(X_1 Y_2)$ is an equilibrium pair.

(*) A mixed strategy X for a player A is called optimal strategy for A if there exists a mixed strategy Y for player B such That $(X Y)$ is an equilibrium pair. Likewise for B.

$S_m = \{(x_1, x_2, \ldots x_m) \mid x_i \geq 0 \ \Sigma x_i = 1\}$

$S_n = \{(y_1, y_2 \ldots y_n) \mid y_i \geq 0 \ \Sigma y_i = 1\}$.

* Given a mixed strategy $X = (x_1, x_2 \ldots x_m)$ for A

define $u(X) = \min_{Y \in S_n} E(X, Y)$.

$u(X) \rightarrow$ Minimum payoff that player A can be guaranteed using strategy X.

why? min exists.

S_n is closed, bounded subset of $\mathbb{R}^n$.

and for a fixed $x_1, x_2 \ldots x_m$.

the function $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$ is continuous on $\underline{S_n}$

so this function has minimum value on S_n.

(*) Given a mixed strategy $Y = (y_1, y_2 \ldots y_n)$ for B define $w(Y) = \max_{X \in S_m} E(X Y)$

$w(Y) \rightarrow$ Maximum payoff that player A hope to get when player B uses strategy Y.

Maximum exist : S_m is closed, bounded.

for fixed $y_1 y_2 \ldots y_n$

$\Sigma \Sigma a_{ij} x_i y_j$ is continuous on S_m

So maximum occurs on $\underline{S_m}$.

Th^m: 3 For any X and Y, $u(X) \leq w(Y)$.

Proof: $u(X) = \min_{Y \in S_n} E(XY) \leq E(X,Y)$ for any mixed strategy Y for B.

Thus $u(X) \leq E(X,Y) \leq \max_{X \in S_m} E(XY) = w(Y)$

connection between equilibrium pair and $u(X)$, $w(Y)$.

Th^m: 4 For any X^* and Y^*: $(X^* Y^*)$ is an equilibrium pair iff $u(X^*) = w(Y^*)$.

Moreover for any equilibrium pair $(X^* Y^*)$

$u(X^*) = w(Y^*) = E(X^* Y^*) = v$.

Pf: First suppose that $(X^* Y^*)$ is equilibrium pair.

Then $E(X^* Y^*)$ is the max^m of $E(X, Y^*)$ $X \in S_m$

ie $w(Y^*) = E(X^* Y^*)$.

slly $E(X^* Y^*)$ is $= \min_{Y \in S_n} E(X^*, Y) = u(X^*)$.

So $u(X^*) = w(Y^*) = E(X^* Y^*)$

conversely assume that $u(X^*) = w(Y^*)$.

$E(X^* Y^*) \leq w(Y^*) = u(X^*) = \min_{Y \in S_n} E(X^* Y) \leq E(X^* Y)$ $\forall Y$.

$E(X^* Y^*) \geq \min_{Y \in S_n} E(X^* Y) = u(X^*) = w(Y^*) = \max_{X \in S_m} E(X Y^*) \geq E(X Y^*)$ $\forall X$

So $(X^* Y^*)$ is an equilibrium pair.

Th^m: 5 Let $(X^* Y^*)$ be an equilibrium pair. Then
the following holds:

(i) $u(X^*) \geq u(X)$ for all X

(ii) $w(Y^*) \leq w(Y)$ for all Y.

Pf: By previous th m^s: we have

$$u(X) \leq w(Y^*) = u(X^*) \quad (\text{By } th^m \, 3 \, \& \, th^m 4)$$

Also $w(Y) \geq u(X^*) = w(Y^*)$

Corollary: For any mixed strategy X for player A and any mixed strategy Y for player B,

$$u(X) \leq v \leq w(Y) \text{ ; where } v \text{ is the value of the game.}$$

Pf: We know that $u(X^*) = w(Y^*) = v$.

By Th^m 5: $\underline{u(X)} \leq u(X^*) = \underline{v} = w(Y^*) \leq \underline{w(Y)}$

Th^m 6: $u(X) = \min\limits_{j=1,2\cdots n} E(X, B_j) = \min\{E(X,B_1), E(X,B_2), \cdots E(X,B_n)\}$

where B_j for $j = 1\,2 \cdots n$ are pure strategies for A.

Pf: Let player A has strategies $A_1, A_2 \cdots A_m$ and player B has strategies $B_1, B_2, \cdots B_n$.

for any mixed strategy $X = (x_1\, x_2 \cdots x_m)$ for player A we know that —

$$u(X) = \min_{Y \in S_n} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j \right\}, \text{~~where~~}$$

$$= \min_{Y \in S_n} \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} a_{ij} x_i \right).$$

This minimum is attained when we let $y_j = 1$ for that j which gives the smallest value for $\sum_{i=1}^{m} a_{ij} x_i$ and 0 for the other coordinates.

Thus $u(X) = \min_{j=1,2,\dots n} \sum_{i=1}^{m} a_{ij} x_i$

$$= \min_{j=1,2,\dots n} E(X, B_j)$$

By a similar argument, we get the following theorem.

Th^m 7: $w(Y) = \max_{i=1,2,\dots m} E(A_i, Y)$

Example: Let $X = (.5, .5, 0)$ and $Y = (.3, .5, .2)$ be mixed strategies for player A and B in the two person zero-sum game given by the payoff matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$

Determine whether $(X\,Y)$ is an equilibrium pair.

Sol^n: Let A_1, A_2, A_3 be player A's Strategies and B_1, B_2, B_3 be the player B's strategies. (8)

Then $E(X B_1) = 3.5$, $E(X, B_2) = 1$, $E(X B_3) = 1.5$

Thus $v(X) = \min\{3.5, 1, 1.5\} = 1$.

Similarly, $E(A_1, Y) = 1.8$, $E(A_2, Y) = 1.9$, $E(A_3 Y) = 1.6$

Thus $w(Y) = \max\{1.8, 1.9, 1.6\} = 1.9$.

We see that $v(X) \neq w(Y)$.

Hence (X, Y) is not an equilibrium pair.

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